Collective modes of trapped gases at the BEC-BCS crossover

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The collective mode frequencies in isotropic and deformed traps are calculated for general polytropic equation of states, $P \propto n^{\gamma+1}$, and expressed in terms of γ and the trap geometry. For molecular and standard Bose-Einstein condensates and Fermi gases near Feshbach resonances, the effective power $\gamma \simeq 0.5 - 1.3$ is calculated from Jastrow type wave-function ansätze, and from the crossover model of Leggett. The resulting mode frequencies are calculated for these phases around the BCS-BEC crossover.

Recent experiments probe systems of fermions [1–5] and bosons [6] near Feshbach resonances by expansion and RF spectroscopy. Interesting new strongly interacting or dense phases of bosons and fermions are created, e.g., that associated with the crossover from a (possibly superfluid) Fermi gas to a molecular BEC. The corresponding equations of states (EOS) differ from standard dilute systems which directly shows up in their collective modes.

The purpose of this work is to calculate the collective modes in terms of a general class of polytropic EOS, to calculate the EOS for strongly interacting BEC and Fermi gas at the crossover to a molecular BEC, and finally synthesize the two to calculate the collective modes for these strongly interacting phases.

Polytropic EOS relate the pressure and density as

$$P \propto n^{\gamma+1}. \qquad (1)$$

As we shall show below the collective modes in harmonic oscillator traps depend on the power γ but not on other details of the polytropic EOS. Polytropic EOS apply to many systems. In a dilute interaction dominated BEC $\gamma = 1$, whereas an ideal Bose gas in the normal state has $\gamma = 2/3$ under adiabatic conditions. A dilute gas of Fermi atoms also has $\gamma = 2/3$ in both the hydrodynamic and superfluid limits. Both a Fermi gas [7] and a BEC [8] has $\gamma = 2/3$ in the strongly interacting (unitarity) limit. We shall see below that near a Feshbach resonance, where a Fermi gas crossover to a molecular BEC, the power effectively varies between $\gamma \sim 0.5 - 1.3$.

The collective modes are calculated from the equations of motion which in hydrodynamics and for a superfluid are given by the equation of continuity and the Euler equation

$$mn\frac{\partial \mathbf{v}}{\partial t} = -\nabla P - n\nabla V_{ext}. \qquad (2)$$

Here n and $\mathbf{v}$ are the local density and velocity, and $V_{ext} = (1/2)m\sum_i \omega_i^2 r_i^2$ is the harmonic oscillator trap potential. From the Euler equation we obtain the equilibrium density: $n_{eq} = n_0(1 - \sum_i r_i^2/R_i^2)^{1/\gamma}$, where $R_i^2 = 2(\gamma+1)P_0/\gamma n_0 m\omega_i^2$, $i = 1, 2, 3$, are the 3D Thomas-Fermi radii of the trapped cloud of atoms (P_0 and n_0 are the pressure and density in the center of the trap).

Linearizing around equilibrium, $n = n_{eq} + e^{i\omega t}\delta n$, the equations of motion lead to

$$-m\omega^2\delta n = \nabla \cdot \left[n\nabla\left(\frac{1}{n}\frac{dP}{dn}\delta n\right)\right]. \qquad (3)$$

It is not necessary to restrict ourselves to zero temperature where the Gibbs-Duhem relation $dP = nd\mu$ simplifies Eq. (3).

In an isotropic trap the collective modes with angular momentum l and n radial nodes are straight forward to calculate from Eq. (3) by generalizing the method of Ref. [9] to any polytropic EOS. We find that the departure from the equilibrium density is $\delta n \propto r^l(1 - r^2/R^2)^{(1/\gamma-1)}F(-n, n+l+\gamma^{-1}, l+3/2, r^2/R^2)$, where F is the hypergeometrical function. The corresponding eigenvalues are

$$\frac{\omega^2}{\omega_0^2} = l + 2n[\gamma(n+l+1/2)+1], \qquad (4)$$

which reduces to the known results for $\gamma = 1$ [10] and $\gamma = 2/3$ [11]. In comparison the collective modes in the collisionless limit are those of a free particle: $\omega/\omega_0 = 2n + l$, when its mean free path exceeds the size of the cloud.

The hydrodynamic collective modes can also be calculated for deformed traps for a general polytropic EOS. Linearizing the equations of motion lead to the following equation for the collective modes

$$-\omega^2\mathbf{v} = \nabla(\mathbf{v} \cdot \nabla V_{ext}) + \gamma(\nabla V_{ext})(\nabla \cdot \mathbf{v}). \qquad (5)$$

The breathing modes have flow velocity on the form $\mathbf{v} = (a_1 r_1, a_2 r_2, a_3 r_3)e^{i\omega t}$, which leads to three coupled homogeneous equation for $a_{i=1,2,3}$

$$\gamma\sum_j \omega_j^2 a_j = (\omega^2 - (\gamma+2)\omega_i^2)a_i. \qquad (6)$$

In an axial symmetric trap: $\omega_1 = \omega_2 \equiv \omega_0$ and $\omega_3 = \lambda\omega_0$, the resulting breathing modes are [12]

$$\frac{\omega^2}{\omega_0^2} = \gamma + 1 + \frac{\gamma+2}{2}\lambda^2$$
$$\pm \sqrt{(\gamma+2)^2\lambda^4/4 + (\gamma^2 - 3\gamma - 2)\lambda^2 + (\gamma+1)^2}, \quad (7)$$

and $\omega = \sqrt{2}\omega_0$. The $\pm$ eigenvalues are the radial and axial modes respectively and result from the coupled monopole and quadrupole $m = 0$ modes, where m is the angular momentum projection on the 3rd axis. Therefore, the breathing modes for an isotropic trap $\lambda = 1$ become the quadrupole with $\omega = \sqrt{2}\omega_0$ and the monopole with $\omega = \sqrt{3\gamma + 2}\omega_0$ as follows from both Eq. (7) and Eq. (4) for $n = 0, l = 2$ and $n = 1, l = 0$ respectively. For $\gamma = 1$ and $\gamma = 2/3$ the monopole frequencies are the standard $\omega = \sqrt{5}\omega_0$ and $\omega = 2\omega_0$ respectively.

For a very elongated or cigar-shaped trap (prolate in nuclear terminology), $\lambda \ll 1$, used in recent experiments [5], Eq. (7) results in a low frequency axial mode with

$$\omega_{ax} = \sqrt{3 - (\gamma + 1)^{-1}}\,\omega_3\,. \qquad (8)$$

For $\gamma = 1$ and $\gamma = 2/3$ the axial mode frequencies are $\omega = \sqrt{5/2}\omega_3$ and $\omega = \sqrt{12/5}\omega_3$ respectively as found in [13,14]. The radial modes have

$$\omega_{rad} = \sqrt{2(\gamma + 1)}\,\omega_0\,. \qquad (9)$$

For $\gamma = 1$ and $\gamma = 2/3$ the radial mode frequencies are the standard $\omega = 2\omega_0$ and $\omega = \sqrt{10/3}\omega_0$ respectively.

In the oblate limit, $\lambda \gg 1$, the breathing modes are $\omega_{ax} = \sqrt{\gamma + 2}\omega_3$ and $\omega_{rad} = \sqrt{(6\gamma + 4)/(\gamma + 2)}\omega_0$. They connect to the prolate limit through avoided level crossing as seen in Fig. 1.
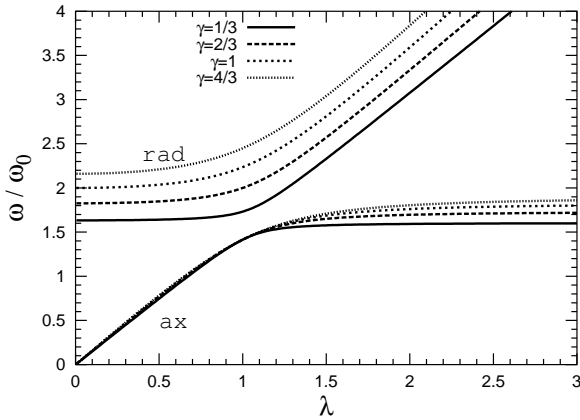

FIG. 1. The $\pm$ solutions of Eq. (7) vs. trap deformation $\lambda = \omega_3/\omega_0$ for $\gamma = 1/3, 2/3, 1, 4/3$. For a cigar shaped trap, $\lambda \ll 1$, these correspond to the radial and axial mode frequencies respectively.

We now turn to the EOS for strongly interacting Bose and Fermi gases and calculate an effective polytropic index that can be applied for the above modes. The EOS for a BEC was calculated in Ref. [8] from a Jastrow type wave function $\Psi_J(\mathbf{r}_1, ..., \mathbf{r}_N) = \prod_{i<j} f(\mathbf{r}_i - \mathbf{r}_j)$, which incorporates essential two-body correlations and is a good approximation for cold dilute and dense bose systems [15]. It was shown that with proper boundary conditions the calculated energy reproduced the dilute limit result, $E/N = 2\pi\hbar^2 an/m$, where a is the s-wave scattering length between bosons. However, in the unitarity limit, $n^{1/3}a \gg 1$ or $x = 1/ak_F \simeq 0$, the energy per particle scales like a Fermi gas polytrope: $E/N = 13.33\hbar^2 n^{2/3}/m = 2.79E_F$. Here, we have also for bosons defined $E_F = \hbar^2 k_F^2/2m$ in terms of the density $n = k_F^3/3\pi^2$ for later comparison between molecular BEC and Fermi gases with two spin states. In this model (see [8,16] and Fig. 2 for details) we can calculate the zero temperature pressure $P = n^2 d(E/N)/dn$ and the effective polytropic index, which we define as the logarithmic derivative

$$\bar{\gamma} \equiv \frac{n}{P}\frac{dP}{dn} - 1\,. \qquad (10)$$

As shown in Fig. 3 $\bar{\gamma}$ approaches 1 asymptotically from above in the dilute limit but is $2/3$ in the unitarity limit with a maximum of ~ 1.25 around $1/k_F a_m \sim 1.4$.
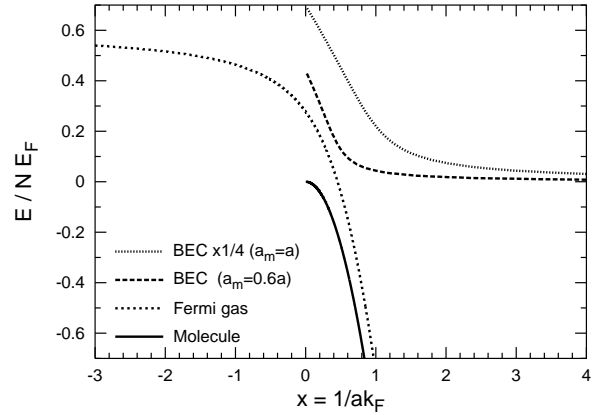

FIG. 2. The energy per particle in units of E_F for a BEC with $a_m = a$ and $a_m = 0.6a$, and a Fermi gas as it crossover towards a molecular BEC (see text). At small positive scattering length the energy per fermion approaches half the binding energy of a molecule, $E/N \to -\hbar^2/2ma^2 = -x^2 E_F$.

The approximate EOS can also be applied to a molecular BEC when a number of factors are taken into account: the density of molecules is half that of Fermi atoms, their mass is two times larger, the above calculated E/N is for two atoms, and finally the scattering length between molecules (a_m) may differ from that between the two Fermi atoms forming the molecule. Petrov et al. find [17] $a_m = 0.6a$ in accordance with [3]. As a result the energy in molecular BEC is per atom much smaller than that of a BEC of Bose atoms as seen in Fig. 2. This is important when we attempt to match the EOS of a molecular BEC onto a gas of Fermi atoms in the unitarity limit. The difference in magnitude cancels in Eq. (10) and is therefore not important for calculating $\bar{\gamma}$. A difference between a and a_m and between Bose and Fermi densities does, however, affect ak_F as seen in Fig. 3.

The EOS for a Fermi gas at zero temperature near a Feshbach resonance and its crossover to a molecular BEC has recently been studied by various resummation techniques [18,7], a Jastrow-Slater type ansatz [7,16] and

by fixed-node Greens function Monte Carlo (FN-GFMC) [19]. The EOS calculated from the Jastrow-Slater ansatz has the merit that the EOS is exact to leading orders both in the dilute and molecular limits. Furthermore, it has been tested experimentally [16] and in FN-GFMC to be a good approximation in the unitarity limit as well. It extends the Jastrow wave function for bosons described above by including an anti-symmetric Slater wave function (Φ_S), which is the ground state of free fermions $\Psi_{JS}(\mathbf{r}_1,...,\mathbf{r}_N) = \Phi_S \prod_{i<j'} f(\mathbf{r}_i - \mathbf{r}_{j'})$. Because Φ_S insures that same spins are spatially anti-symmetric, the Jastrow wave function only applies to particles with different spins (indicated by the primes). The resulting energy (see [19,16] for details) is shown in Fig. 2. The energy per particle at zero temperature can generally be written in terms of the ratio between the interaction and kinetic energies $\beta = E_{int}/E_{kin}$ as [7,1]

$$E/N = E_{kin} + E_{int} = \frac{3}{5}E_F[1 + \beta]. \qquad (11)$$

As seen in Fig. 2 it approaches the ideal Fermi gas result $(3/5)E_F$ in the dilute limit, $(3/5)(1+\beta)E_F$ in the unitarity limit with $\beta \simeq -0.54$ for two spin states [16], and $E/N = -\hbar^2/2ma^2$ in the molecular BEC limit ($a \to 0_+$). The corresponding pressure is

$$P = n^2 \frac{dE/N}{dn} = \frac{2}{5}E_F n[1 + \beta - x\beta'/2]. \qquad (12)$$

Here, we view $\beta(x)$ as a smooth function of $x = 1/ak_F$ with derivatives $\beta' = d\beta(x)/dx$, etc. The effective polytropic index is from Eqs. (10) and (12)

$$\bar{\gamma} = \frac{\frac{2}{3}(1+\beta) - x\beta'/2 + x^2\beta''/6}{1 + \beta - x\beta'/2}. \qquad (13)$$

When $\beta(x)$ is a smooth function at $x = 0$ we find from Eq. (13) that $\bar{\gamma} = 2/3$ in the unitarity limit.

The effective polytropic index is shown in Fig. 3 for a Fermi gas as it crossover to a molecular BEC. That it turns over and drops back to $\gamma \to 2/3$ for $x \gtrsim 0.5$ is an artifact of the EOS resulting from the Slater ansatz in the wave function. The true ground state wave function is expected to have a lower energy in the molecular BEC limit as $a \to 0$ as is also found in FN-GFMC [19]. The Jastrow part of the wave function is responsible for the correct leading part of the energy: $E/N = -\hbar^2/2ma^2$ which, however, does not contribute to the pressure because it is density independent. The Slater wave function is responsible for the leading density dependent order but detailed comparison to FN-GFMC calculations show that it is only correct up to $x \lesssim 0.5$. In FN-GFMC a better ground state wave function is found numerically which has lower energy. Both Fermi gases and BEC's have $\gamma = 2/3$ at $x = 0$ due to the universal scaling law $E/N \propto n^{2/3}$ in the unitarity limit [7,8].

Leggett [20] extended the BCS gap equation to the BCS-BEC crossover and calculated the gap and chemical potential. From the Gibbs-Duhem relation and Eq.

(10) we can then calculate $\bar{\gamma}$ as shown in Fig. 3. In the dilute BCS limit it differs from the Fermi gas, which has chemical potential $\mu = E_F + 2\pi\hbar^2 an/m$, by lacking this second term proportional to a. In the other limit the chemical potentials differ for orders higher than linear in the scattering length. In both the dilute BEC and the Jastrow BEC approximation higher orders add positively - which is responsible for $\bar{\gamma} > 1$ for $k_F a_m \gtrsim 1$. The Legget model leads to negative higher order contributions beyond the linear one: $\mu = -\hbar^2/2ma^2 + \pi\hbar^2 an/m$, and therefore $\bar{\gamma}$ decreases monotonously from 1 towards 2/3 as $x \to 0_+$.

In all the above models the two-body correlation function undergoes a smooth transition from a constant in the dilute Fermi gas limit to that of a molecule in the dilute molecular Bose gas limit, and the BCS-BEC crossover is continuous. In the unitarity region the correlation length is of order the interparticle spacing $\sim k_F^{-1}$; only in the dilute BEC limit $x \gg 1$ is the correlation length sufficiently small that the molecules may be approximated as point particles.
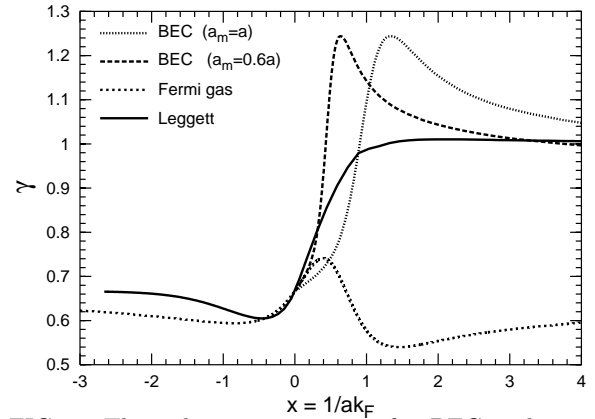

FIG. 3. The polytropic power $\bar{\gamma}$ of a BEC with $a_m = a$ and $a_m = 0.6a$, a Fermi gas (Jastrow-Slater ansatz), and the Leggett model describing the smooth crossover to a molecular BEC. See text and Fig. 2.

The strongly interacting EOS's near the unitarity limits can be approximated by a polytrope by replacing γ with $\bar{\gamma}$. This allows us to calculate the collective modes directly from Eqs. (4) and (7) as shown in Fig. 4. Experimentally one tunes the scattering length near a Feshbach resonance for a fixed number of trapped particles N whereby the size and density of the cloud and therefore also k_F varies in a complicated way depending on the EOS. For fermions in the dilute and unitarity limit $\bar{\gamma} = 2/3$ and the size of cloud is $R = (24N)^{1/6}a_{osc}(1+\beta)^{1/4}$ with $\beta = 0$ and $\beta \simeq -0.56$ respectively. Since $\bar{\gamma} \simeq 2/3$ for a Fermi gas up to and around the unitarity limit this relation for the size is a good approximation in this region, and analogously: $k_F \simeq (24N)^{1/6}a_{osc}^{-1}(1+\beta)^{1/4}$. In a dilute BEC: $R = (15Na_m)^{1/5}a_{osc}^{4/5}$ and $k_F = (3\pi/8)^{1/3}(15N)^{2/15}a_m^{-1/5}a_{osc}^{-4/5}$ in the center of the trap.

The resulting collective modes - in particular the radial

breathing frequency - are as shown in Fig. 4 very sensitive to $x = 1/ak_F$ through γ near the unitarity limit, where the Fermi gas crossover to a molecular BEC. The modes can therefore be exploited to extract the EOS experimentally. The leading corrections in the dilute Fermi gas and BEC were given in [21]. It is observed from Fig. 4 that the frequencies are very sensitive to a_m/a and may therefore be exploited to relate the atomic and molecular scattering lengths.
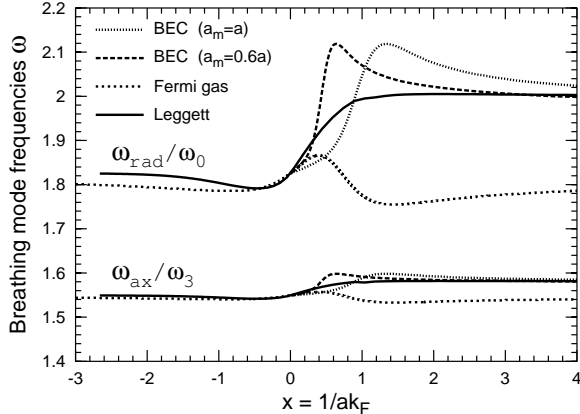

FIG. 4. The radial and axial frequencies for cigar shaped traps, $\lambda \ll 1$, from Eqs. (8) and (9) with γ from from Fig. 3.

In very dilute Fermi gases, where the pairing gap becomes smaller than the oscillator frequency and the coherence length exceeds the system size, particle excitations of order $\sim 2\Delta$ appear [11,22] besides the collective modes described above. Using the pairing gap in an isotropic dilute trap [22] the condition $\Delta < \hbar\omega_0$ becomes $x = 1/ak_F \lesssim -(2/\pi)(C + \ln(3N)/3)$, where $C = 0.577..$ is Eulers constant. Such pair excitation modes are therefore only observable for weak attractions.

The collective mode frequencies do not distinguish between a superfluid and a hydrodynamic Fermi gas. The damping of the modes should be different but has not been estimated in the unitarity limit for bosons or fermions. Assuming an unitarity limited scattering cross section we expect the collision rate to decrease as $\sim exp(-\Delta(T)/T)$ at temperatures well below the gap $\Delta(T = 0) \simeq 0.54 E_F \exp(\pi x/2)$ [19,16] in a bulk system. The damping can potentially discriminate between hydrodynamic and superfluid Fermi gases, and at the same time the mode frequency discriminates collisionless.

In summary, the dependence of collective mode frequencies and damping on density, interaction strength and temperature as described above can - especially near the BCS-BEC crossover - reveal the underlying EOS in detail including possible phase transitions and associated critical temperatures and densities.

Note added in proof: Two experiments have recently published the axial and radial modes around the unitarity limit. The above results for the Leggett model is in nice agreement with the data of Kinast et al. [23] and (for the axial mode) with Bartenstein et al. [24]. See also [25] for

details. The Leggett model also explains "surprise one" in [24]

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